CSCC37 Linear Systems Notes

1. Linear Algebra Review:

a) Terminology:

Let A be a $m \times n$ matrix:

- This means that A has m rows and n colns.
- If $m=n$, then A is a square matrix.

Let B be a $n \times n$ matrix:

- B is a square matrix.
- B is said to be singular if it has 1 of the following equivalent properties:

1. B has no inverse.
2. $\det(B)=0$
3. $B\bar{z}=\bar{0}$ for some vector $\bar{z} \neq \bar{0}$

Otherwise, B is non-singular. If B is non-singular, then B^{-1} exists and the system $B\bar{x}=\bar{b}$ always has a unique soln $\bar{x}=B^{-1}\bar{b}$ regardless of the value of $\bar{b}$. If B is singular, it will either have no solns or infinitely many solns.

- The main diagonal of B is the values $B_{11}, B_{22}, \ldots, B_{nn}$.

E.g. Let $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

The main diagonal is circled in red.

General Terminology:

- The transpose of matrix $A_{m,n}$, denoted as A^T, is created when you switch the row and coln indices of each element in A.

E.g. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ $A^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$

If A is a $m \times n$ matrix, A^T is a $n \times m$ matrix.

To create A^T from A, write the rows of A as the colns of A^T.

A square matrix whose transpose is equal to itself is a symmetric matrix.
I.e. If $A^T = A$, then A is a symmetric matrix.

A square matrix whose transpose is equal to its negative is a skew-symmetric matrix.
I.e. If $A^T = -A$, then A is a skew-symmetric matrix.

- The identity matrix, denoted as I, is a square matrix with 1's along the main diagonal and 0 elsewhere.
E.g. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are identity matrices.

Note: The identity matrix is a symmetric matrix.
Note: The product of 2 inverse matrices is always the identity matrix.
I.e Let $B = A^{-1}$. Then, $AB = BA = I$

- A lower triangular matrix is a square matrix if all entries above the main diagonal is 0.
E.g. $A = \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix}$ is a lower triangular matrix.

- An upper triangular matrix is a square matrix if all entries below the main diagonal is 0.
E.g. $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ is an upper triangular matrix.

- A **permutation matrix**, denoted as P, is a square matrix having exactly one 1 in each row and coln and 0 elsewhere. It is used if you want to swap 2 rows.

E.g. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Suppose you want to swap rows 1 and 2.
Your permutation matrix, P, would be $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$$

E.g. Let $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

Suppose you want to swap rows 1 and 3.
Your permutation matrix P would be $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 7 & 8 & 9 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix}$$

Now suppose you want to swap rows 2 and 3 of the original matrix. Your P would be $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \\ 4 & 5 & 6 \end{bmatrix}$$

Suppose you want to swap rows i and j, $i \neq j$. To create/determine your permutation matrix, start with the identity matrix. Then, move the 1 at position (i,i) to (i,j) and move the 1 at position (j,j) to position (j,i).

In the first example, when I wanted to swap rows 1 and 2, $i=1$ and $j=2$. The 1 at $(1,1)$ got moved to $(1,2)$ and the 1 at $(2,2)$ got moved to $(2,1)$.

b) Calculations:

Matrix Addition and Subtraction:

- Two matrices, A and B can only be added or subtracted if they have the same number of rows and colns.
- $$A \pm B = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \pm \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{m1} & \cdots & b_{mn} \end{bmatrix}$$

E.g. 1 Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 5 \\ 1 & 5 \end{bmatrix}$

Find $A+B$ and $A-B$

Soln:

$$A+B = \begin{bmatrix} 1+3 & 2+5 \\ 3+1 & 4+5 \end{bmatrix} = \begin{bmatrix} 4 & 7 \\ 4 & 9 \end{bmatrix}$$

$$A-B = \begin{bmatrix} 1-3 & 2-5 \\ 3-1 & 4-5 \end{bmatrix} = \begin{bmatrix} -2 & -3 \\ 2 & -1 \end{bmatrix}$$

Matrix Multiplication:

- We can only multiply 2 matrices, A and B, if the number of colns of A = the num of rows of B. The resulting matrix will have the same number of rows as A and the same number of colns as B.

E.g. 2 Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 1 & 2 \\ 5 & 2 & 3 \end{bmatrix}$

Find $A \times B$

Soln:

$$A \times B = \begin{bmatrix} (1)(3)+(2)(5) & (1)(1)+(2)(2) & (1)(2)+(2)(3) \\ (3)(3)+(4)(5) & (3)(1)+(4)(2) & (3)(2)+(4)(3) \\ (5)(3)+(6)(5) & (5)(1)+(6)(2) & (5)(2)+(6)(3) \end{bmatrix}$$

$$= \begin{bmatrix} 13 & 5 & 8 \\ 29 & 11 & 18 \\ & 17 & 28 \end{bmatrix}$$

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2. Linear Systems:

- $A\bar{x} = \bar{b}$, where $A \in \mathbb{R}^{n \times n}$, $\bar{x}, \bar{b} \in \mathbb{R}^n$

We're given A and $\bar{b}$ and have to solve for $\bar{x}$.

- General Soln Technique:

1. Reduce the problem to an equivalent one that's easier to solve.
2. Solve the reduced problem.

E.g. 3 Solve $3x_1 - 5x_2 = 1$
$6x_1 - 7x_2 = 5$

Soln:

$$\begin{bmatrix} 3 & -5 \\ 6 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix} \begin{matrix} r_1 \\ r_2 \end{matrix}$$

Do $r_2 - 2r_1$

$$\begin{bmatrix} 3 & -5 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$3x_2 = 3 \Rightarrow x_2 = 1$
$6x_1 - 7x_2 = 5$
$6x_1 - 7 = 5$
$6x_1 = 12$
$x_1 = 2$

$x_1 = 2,\ x_2 = 1$

Now, we'll generalize this to n unknowns and n eqns.
Let matrix $A = [a_{ij}]$ where a_{11} is the top left element.

Eqn 1: $a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$
Eqn 2: $a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$
$\vdots$
Eqn n: $a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n$

We will use the following steps to reduce this system to triangular form.

Recall: An upper triangular matrix is a square matrix with 0's below the main diagonal.

Step 1:

- Assume that $a_{11} \neq 0$
- Multiply Eqn 1 by $\frac{a_{21}}{a_{11}}$ and subtract from Eqn 2.
- Multiply Eqn 1 by $\frac{a_{31}}{a_{11}}$ and subtract from Eqn 3.
- Repeat for all remaining rows.

I.e. Multiply Eqn 1 by $\frac{a_{i1}}{a_{11}}$ and subtract from Eqn i where $4 \leq i \leq n$.

- We now have an equivalent system where x_1 has been eliminated from eqns 2 to n.

I.e Now we have:

$$
\begin{array}{llllll}
\text{Eqn } 1: & a_{11}x_1 & + a_{12}x_2 & + \dots + & a_{1n}x_n & = b_1 \\
\text{Eqn } \hat{2}: & 0 & + \hat{a}_{22}x_2 & + \dots + & \hat{a}_{2n}x_n & = \hat{b}_2 \\
\vdots & \vdots & & & & \\
\text{Eqn } \hat{n}: & 0 & + \hat{a}_{n2}x_2 & + \dots + & \hat{a}_{nn}x_n & = \hat{b}_n
\end{array}
$$

Note: The small hat, ^, is used to note that these values changed.

Step 2:

- Assume that $\hat{a}_{22} \neq 0$.
- Multiply eqn 2 by $\frac{\hat{a}_{32}}{\hat{a}_{22}}$ and subtract from Eqn 3.
- Repeat for all remaining rows.
- We now have an equivalent system where x_2 has been eliminated from eqns 3 to n.

Repeat this pattern up to and including eqn n-1. Afterwards, we will have an upper triangular system.

I.e.

Eqn 1: $a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$

Eqn $\hat{2}$: $0 + \hat{a}_{22}x_2 + \dots + \hat{a}_{2n}x_n = \hat{b}_2$

$\vdots$

Eqn $\tilde{n}$: $0 + 0 + \dots + \tilde{a}_{nn}x_n = \tilde{b}_n$

- Another way of looking at this is using vector notation. ($A\bar{x} = \bar{b}$)

Step 1: Eliminate the first coln of A using a_{11}.

$L_1 A\bar{x} = L_1\bar{b}$, where $L_1 = \begin{bmatrix} 1 & 0 & \dots & 0 \\ -\frac{a_{21}}{a_{11}} & 1 & & \\ \vdots & & \ddots & \\ -\frac{a_{n1}}{a_{11}} & & & 1 \end{bmatrix}$

Note: L_1 is very similar to the identity matrix except the first coln is filled with multipliers used in the first step.

Step 2: Eliminate the second coln of L_1A using $\hat{a}_{22}$.

$L_2(L_1A)\bar{x} = L_2(L_1\bar{b})$ where $L_2 = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & & \vdots \\ \vdots & -\frac{\hat{a}_{32}}{\hat{a}_{22}} & & \\ 0 & \vdots & & 1 \\ & -\frac{\hat{a}_{n2}}{\hat{a}_{22}} & & \end{bmatrix}$

We continue until we have $L_{n-1} L_{n-2} \ldots L_2 L_1 A\bar{x} = L_{n-1} L_{n-2} \ldots L_2 L_1 \bar{b}$.

We let $L_{n-1} L_{n-2} \ldots L_1 A = U$ where U is an upper triangular matrix. This becomes very easy to solve.

E.g. 4 Solve the following system of eqns using the technique we just learned.

$$2x_1 + 4x_2 - 2x_3 = 2$$
$$4x_1 + 9x_2 - 3x_3 = 8$$
$$-2x_1 - 3x_2 + 7x_3 = 10$$

Soln:

$$\begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$$

$$L_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$L_1(A) = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 1 & 5 \end{bmatrix}$$

$$L_1 \bar{B} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 12 \end{bmatrix}$$

$$L_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$L_2(L_1 A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 1 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix}$$

$$L_2(L_1 \bar{B}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 12 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix}$$

We now have

$$\begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix}$$

This makes finding $\bar{x}$ much easier.

$$\bar{x} = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$$

3. LU Factorization:

– We have $L_{n-1} L_{n-2} \dots L_1 A = U \Leftrightarrow A = \overbrace{L_1^{-1} L_2^{-1} \dots L_{n-1}^{-1}}^{L} U$

Lemma 1: If L_i is a Gauss Transformation, then L_i^{-1} exists and is also a Gauss Transformation.

Lemma 2: If L_i and L_j are Gauss Transformations and $j>i$, then $L_i L_j = L_i + L_j - I$

– $A = L_1^{-1} L_2^{-1} \dots L_{n-1}^{-1} U \Leftrightarrow A = LU$

– Using $A = LU$ to solve $A\bar{x} = \bar{b}$, we can convert $A\bar{x} = \bar{b}$ into $(LU)\bar{x} = \bar{b}$. Then, let $\bar{d} = U\bar{x}$ where $L\bar{d}$ is a lower triangular matrix. We now have $L\bar{d} = \bar{b}$.

I.e.

$A\bar{x} = \bar{b}$

$\Leftrightarrow LU\bar{x} = \bar{b}$

$\Leftrightarrow L\bar{d} = \bar{b}$ where $\bar{d} = U\bar{x}$.

while $L\bar{d}$ is a lower triangular matrix and $U\bar{x}$ is an upper triangular matrix.

– We use LU factorization because if we have the same coefficient matrix A but different RHS, we can use the same LU.

E.g. 5

Let $A = \begin{bmatrix} 2 & -1 & 1 \\ 2 & 2 & 2 \\ -2 & 4 & 1 \end{bmatrix}$

Find L_1 and L_2

Soln:

$$L_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$L_1 A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ 2 & 2 & 2 \\ -2 & 4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 & 1 \\ 0 & 3 & 1 \\ 0 & 3 & 2 \end{bmatrix}$$

$$L_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$L_2(L_1 A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ 0 & 3 & 1 \\ 0 & 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{bmatrix} \leftarrow U$$

Recall: $L_{n-1} L_{n-2} \dots L_1 A = U$

Now, let's show Lemma 2.
Recall that $L = L_1^{-1} L_2^{-1} \dots L_{n-1}^{-1}$ and that $L_i L_j = L_i + L_j - I$, $j > i$.

To compute L_i^{-1}, simply take L_i and toggle/switch the sign of the multipliers.

$$L_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \rightarrow \quad L_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$L_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \rightarrow \quad L_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\begin{aligned} L &= L_1^{-1} \cdot L_2^{-1} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \end{aligned}$$

Another way to compute $L_1^{-1} \cdot L_2^{-1}$ is $L_1^{-1} + L_2^{-1} - I$

$$\begin{aligned} L_1^{-1} + L_2^{-1} - I &= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ -1 & 1 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \end{aligned}$$

Same as above

Now that we have L and U, lets see what LU is.

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 1 \\ 2 & 2 & 2 \\ -2 & 4 & 1 \end{bmatrix}$$

↑ A

E.g. 6 Given $A = \begin{bmatrix} 8 & 2 & 9 \\ 4 & 9 & 4 \\ 6 & 7 & 9 \end{bmatrix}$

find L and U.

Soln:

$$L_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3/4 & 0 & 1 \end{bmatrix}$$

$$L_1(A) = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ -3/4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 8 & 2 & 9 \\ 4 & 9 & 4 \\ 6 & 7 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 2 & 9 \\ 0 & 8 & -1/2 \\ 0 & \frac{22}{4} & 9/4 \end{bmatrix}$$

$$L_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{11}{16} & 1 \end{bmatrix}$$

$$L_2(L_1 A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{11}{16} & 1 \end{bmatrix} \begin{bmatrix} 8 & 2 & 9 \\ 0 & 8 & -1/2 \\ 0 & \frac{22}{4} & 9/4 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 2 & 9 \\ 0 & 8 & -1/2 \\ 0 & 0 & \frac{83}{32} \end{bmatrix} \leftarrow U$$

$$L = L_1^{-1} + L_2^{-1} - I$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/4 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{11}{16} & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/4 & 11/16 & 1 \end{bmatrix}$$

4. GE with Pivoting:

- If you go back to page 7, you'll see that we have "Assume that $a_{11} \neq 0$" and "Assume that $\hat{a}_{22} \neq 0$." But what happens if at some step, $a_{ii} = 0$?

E.g. $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 3 \end{bmatrix}$

$L_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

$L_1 A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 3 \end{bmatrix}$

$= \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 3 \\ 0 & 1 & 2 \end{bmatrix}$ ← Notice that we can no longer divide by $\hat{a}_{22}$ as it is 0.

A possible soln, in this case, is to swap rows 2 and 3.

We now have $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$

- In general, if $a_{ii} = 0$, go down coln i, starting from row $i+1$, and find a suitable row to swap with row i.
Note: If we want to find a row to swap with row i, we can only choose rows that are below row i. That is, we can only choose row j to swap with row i if $j > i$.

- A similar problem arises/occurs if one of the elements along the main diagonal is very small.
This is a more common scenario than the previous one.

E.g. Let $L_1A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 10^{-16} & 3 \\ 0 & 1 & 2 \end{bmatrix}$

$L_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -10^{16} & 1 \end{bmatrix}$

$L_2(L_1 A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -10^{16} & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 10^{-16} & 3 \\ 0 & 1 & 2 \end{bmatrix}$

$= \begin{bmatrix} 1 & 2 & 1 \\ 0 & 10^{-16} & 3 \\ 0 & 0 & \underbrace{2 - 3 \cdot 10^{16}}_{\text{This term will cause a lot of issues.}} \end{bmatrix}$

A possible soln is to go down the coln, starting from the very small element, and to swap rows where the second row has a bigger element in that coln. This is called partial row pivoting.
For this example, swap rows 2 and 3.

E.g. 7 Solve $\begin{bmatrix} 2 & 6 & 6 \\ 3 & 5 & 12 \\ 6 & 6 & 12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 20 \\ 25 \\ 30 \end{bmatrix}$

using row pivoting.

Soln:

Step 1: We want the biggest value in col 1 to be on the main diagonal. Hence, we swap row 1 and 3.

$P_1 A\bar{x} = P_1 \bar{b}$, where $P_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ **Recall:** P_1 is a permutation matrix.

$P_1 A = \begin{bmatrix} 6 & 6 & 12 \\ 3 & 5 & 12 \\ 2 & 6 & 6 \end{bmatrix}$ $P_1 \bar{b} = \begin{bmatrix} 30 \\ 25 \\ 20 \end{bmatrix}$

Step 2: Find L_1

$$L_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ -1/3 & 0 & 1 \end{bmatrix}$$

$$L_1(P_1 A) = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ -1/3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 6 & 6 & 12 \\ 3 & 5 & 12 \\ 2 & 6 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 6 & 12 \\ 0 & 2 & 6 \\ 0 & 4 & 2 \end{bmatrix}$$

$$L_1(P_1 \bar{b}) = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ -1/3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 30 \\ 25 \\ 20 \end{bmatrix} = \begin{bmatrix} 30 \\ 10 \\ 10 \end{bmatrix}$$

Step 3: We want the biggest value in col 2, Starting, from row 2 to be on the main diagonal. Hence, we swap rows 2 and 3.

$P_2 L_1 P_1 A \bar{x} = P_2 L_1 P_1 \bar{b}$, where $P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$$P_2 L_1 P_1 A = \begin{bmatrix} 6 & 6 & 12 \\ 0 & 4 & 2 \\ 0 & 2 & 6 \end{bmatrix}$$

$$P_2 L_1 P_1 \bar{b} = \begin{bmatrix} 30 \\ 10 \\ 10 \end{bmatrix}$$

Step 4: Find L_2

$$L_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1/2 & 1 \end{bmatrix}$$

$$L_2 P_2 L_1 P_1 A = \begin{bmatrix} 6 & 6 & 12 \\ 0 & 4 & 2 \\ 0 & 0 & 5 \end{bmatrix}$$

$$L_2 P_2 L_1 P_1 \bar{b} = \begin{bmatrix} 30 \\ 10 \\ 5 \end{bmatrix}$$

Now, we have $\begin{bmatrix} 6 & 6 & 12 \\ 0 & 4 & 2 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 30 \\ 10 \\ 5 \end{bmatrix}$ $\bar{x} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

- We have $L_2 P_2 L_1 P_1 A = U$.

$$L_2 P_2 L_1 P_1 A \Longleftrightarrow L_2 P_2 L_1 P_2 P_2 P_1 A \quad (1)$$
$$\Longleftrightarrow L_2 \hat{L}_1 P_2 P_1 A \quad (2)$$

Note: The inverse of a permutation matrix is itself. Hence, that's why we can multiply $L_2 P_2 L_1 P_1 A$ by $P_2 \cdot P_2$ in (1).

Note: $\hat{L}_1 = P_2 L_1 P_2$ is a modified Gauss Transformation.

E.g. 8 Let $P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ and $L_1 = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & 0 & 1 \end{bmatrix}$

Solve $P_2 L_1 P_2$

Soln:

$$P_2 L_1 P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ l_{31} & 0 & 1 \\ l_{21} & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ l_{31} & 1 & 0 \\ l_{21} & 0 & 1 \end{bmatrix}$$

Note: When you pre-multiply a matrix with a permutation matrix, you switch 2 rows. However, when you post-multiply a matrix with a permutation matrix, you switch 2 columns.

I.e. $PL \rightarrow$ Changes 2 rows of L.
$LP \rightarrow$ Changes 2 cols of L.

Hence, $\hat{L}_1$ is L_1 with its 2 multipliers swapped.

Now we have $L_2 \hat{L}_1 P_2 P_1 A = U$
$\Leftrightarrow P_2 P_1 A = \hat{L}_1^{-1} \hat{L}_2^{-1} U$
$\Leftrightarrow PA = LU$
where $P = P_2 P_1$ and $L = \hat{L}_1^{-1} L_2^{-1}$

- Now, we have to solve $A\bar{x} = \bar{b}$ given $PA = LU$.
$A\bar{x} = \bar{b}$
$\Leftrightarrow PA\bar{x} = P\bar{b}$
$\Leftrightarrow LU\bar{x} = \hat{b}$ where $\hat{b} = P\bar{b}$

Let $\bar{d} = U\bar{x}$.
Hence, we solve:
1. $L\bar{d} = \hat{b}$ for $\bar{d}$ (Forward Solve)
2. $U\bar{x} = \bar{d}$ for $\bar{x}$ (Backward Solve)

- What happens if at the k^{th} step, one element along the main diagonal, a_{kk}, and everything below it is 0?

I.e. $$L_{k-1} \ldots L_1 A = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & \ddots & & & \\ & & & 0 & & \\ & & & \vdots & \ddots & \\ & & & 0 & & 1 \end{bmatrix}$$

↑ k^{th} column

Remember that our goal is to make every element in the k^{th} coln under k 0, so we just continue.

However, this will result in a matrix U with a 0 for one of the entries along the main diagonal. Then, we will have a singular matrix U. If U is singular, when we do $U\bar{x} = \bar{d}$, we could have either 0 solns or infinitely many solns.

E.g. $U\bar{x} = \bar{d}$, where $U = \begin{bmatrix} 2 & 5 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

$$\begin{bmatrix} 2 & 5 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

So, we have $x_3 = d_2$
$2x_3 = d_3$

If d_3 is twice d_2, then x_2 is a free variable and we have infinitely many soln.

If d_3 is not twice d_2, then we have no soln.

Note: While it's possible for U to be a singular matrix, L cannot be a singular matrix.

- Now suppose that at the k^{th} step, if all elements below a_{kk} and the element a_{kk} have a magnitude of $\leq eps \cdot \max_{1 \leq j \leq k} |u_{jj}|$. We call this **numerical singularity** or **near singularity**.

5. Complexity of GE:

- Let A be a $n \times n$ matrix.
- We will count additional/multiplication pairs, i.e. $mx+b$, as **Floating Point operation** or **FLOP**.

a) Computing the complexity of LU Factorization:

- Our first step is zeroing out the first col after a_{11}. Hence, we have $(n-1)^2$ FLOPs.

- Our second step is zeroing out the second col after a_{22}. Hence, we have $(n-2)^2$ FLOPs.

$\vdots$

- Our last step is zeroing out the $(n-1)^{th}$ col after $a_{(n-1)(n-1)}$. Hence, we have $(n-(n-1))^2$ or 1 FLOP.

- In total, we have $(n-1)^2 + (n-2)^2 + \ldots + 1$ FLOPs.

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

However, since we go up to $(n-1)^2$, we have $\frac{n(n-1)(2n)}{6}$ or $\frac{n^3}{3} + O(n^2)$ FLOPs for computing the complexity of LU Factorization.

b) Computing the Complexity of Forward and Backward Solve:

- Consider forward solve:

$$\begin{bmatrix} 1 & & & \\ l_{21} & 1 & & 0 \\ l_{31} & & 1 & \\ \vdots & & & \ddots \\ l_{n1} & \cdots & & 1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Recall forward solve is $L\bar{d} = \bar{b}$ where L is a lower triangular matrix.

$d_1 = b_1$ ← No flops
$d_2 = b_2 - l_{21}d_1$ ← 1 flop
$d_3 = \underbrace{b_3 - l_{31}d_1}_{\text{1 flop}} - l_{32}d_2$ ← 2 flops

$\underbrace{\qquad\qquad}_{2^{nd}\text{ flop}}$

$\vdots$
$d_n = (n-1)$ Flops

Hence, we have $0+1+2+\ldots+(n-1)$ FLOPS.

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

However, since we go up to $(n-1)$, we get $\frac{n(n-1)}{2}$

or $\frac{n^2}{2} + O(n)$ FLOPS.

— Backward Solve is similar, resulting in another $\frac{n^2}{2} + O(n)$ FLOPs.

— In total, for forward and backward solve, the complexity is $n^2 + O(n)$ FLOPs.

6. Round off Error:

— Recall that we have $PA = LU$ computed in a floating point system (FPS).

— Because of machine round off error, we actually get $\hat{P}(A+E) = \hat{L}\hat{U}$ where $\hat{P}, \hat{L}, \hat{U}$ are the computed factors and E is the error that occurs during factorization process.

— We hope that E is small relative to A.

— Now, solving $A\bar{x} = \bar{b}$ becomes $(A+E)\hat{\bar{x}} = \bar{b}$ where $\hat{\bar{x}}$ is the computed soln.

— Equivalently, let $E\hat{\bar{x}} = \bar{r}$.
Then, $(A+E)\hat{\bar{x}} = \bar{b}$
$\iff A\hat{\bar{x}} + \bar{r} = \bar{b}$
$\iff \bar{r} = \bar{b} - A\hat{\bar{x}}$, where $\bar{r}$ is the residual. We would like $\bar{r}$ to be $\bar{0}$.

— If we use row partial pivot, we can show that

a) $\|E\| \leq k \cdot eps \cdot \|A\|$ where k is not too large and depends on n.

b) $\|\bar{r}\| \lessapprox k \cdot eps \cdot \|\bar{b}\| \iff \frac{\|\bar{r}\|}{\|\bar{b}\|} \lessapprox k \cdot eps$

$\frac{\|\bar{r}\|}{\|\bar{b}\|}$ is called the **relative residual**.

Note: This does not mean that $\|\bar{x} - \bar{\hat{x}}\|$ or $\frac{\|\bar{x} - \bar{\hat{x}}\|}{\|\bar{x}\|}$ is small.

$\|\bar{x} - \bar{\hat{x}}\|$ is called the absolute error.

$\frac{\|\bar{x} - \bar{\hat{x}}\|}{\|\bar{x}\|}$ is called the relative error.

$\bar{x}$ is the true solution.

E.g. 9. Given $\begin{bmatrix} 0.780 & 0.563 \\ 0.913 & 0.656 \end{bmatrix} \bar{x} = \begin{bmatrix} 0.217 \\ 0.254 \end{bmatrix}$ where

$\bar{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and 2 computed solns $\bar{\hat{x}}_1 = \begin{bmatrix} 0.999 \\ -1.001 \end{bmatrix}$

and $\bar{\hat{x}}_2 = \begin{bmatrix} 0.341 \\ -0.087 \end{bmatrix}$, find $\bar{r}_1$ and $\bar{r}_2$.

$$\bar{r}_1 = \bar{b} - A\bar{\hat{x}}_1 = \begin{bmatrix} -0.001243 \\ -0.001572 \end{bmatrix} \qquad \bar{r}_2 = \bar{b} - A\bar{\hat{x}}_2 = \begin{bmatrix} -0.000001 \\ 0 \end{bmatrix}$$

We see that $\frac{\|\bar{r}_2\|}{\|\bar{b}\|}$ is much smaller than $\frac{\|\bar{r}_1\|}{\|\bar{b}\|}$,

yet we see that $\hat{x}_2$ is a terrible soln.

Furthermore, why is $\frac{\|\bar{x} - \bar{\hat{x}}_1\|}{\|\bar{x}\|}$ so much smaller

than $\frac{\|\bar{x} - \bar{\hat{x}}_2\|}{\|\bar{x}\|}$?

- We need a relationship between relative error and relative residual.

Note: A small residual does not always mean a small error.

We have 2 eqns to start off:

1. $A\hat{\bar{x}} = \bar{b} - \bar{r}$
2. $A\bar{x} = \bar{b}$

We will now subtract (1) from (2) We get:

$A(\bar{x} - \hat{\bar{x}}) = \bar{r}$ (3)

Rearranging (3) by multiplying both sides by A^{-1} gets us:

$\bar{x} - \hat{\bar{x}} = A^{-1}\bar{r}$ (4)

Taking the norm of both sides of (4) gets us:

$$\|\bar{x} - \hat{\bar{x}}\| = \|A^{-1}\bar{r}\|$$
$$\leq \|A^{-1}\| \, \|\bar{r}\| \quad (5)$$

Taking the norm of $\bar{b} = A\bar{x}$, we get

$$\|\bar{b}\| = \|A\bar{x}\|$$
$$\leq \|A\| \, \|\bar{x}\| \quad (6)$$

Combining (5) and (6), we get

$$\frac{\|\bar{x} - \hat{\bar{x}}\|}{\|A\| \, \|\bar{x}\|} \leq \frac{\|A^{-1}\| \, \|\bar{r}\|}{\|\bar{b}\|}$$

$$\underbrace{\frac{\|\bar{x} - \hat{\bar{x}}\|}{\|\bar{x}\|}}_{\text{Relative error}} \leq \frac{\|A\| \, \|A^{-1}\| \, \|\bar{r}\|}{\|\bar{b}\|} = \underbrace{\|A\| \, \|A^{-1}\|}_{\text{Cond}(A)} \, \underbrace{\frac{\|\bar{r}\|}{\|\bar{b}\|}}_{\text{Relative residue}}$$

Note: $1 = \|I\|$
$= \|A \cdot A^{-1}\|$
$\leq \|A\| \, \|A^{-1}\|$
$= \text{Cond}(A)$

$\text{Cond}(A) = \|A\| \, \|A^{-1}\|$. $\text{Cond}(A) \geq 1$ always.

If $\text{cond}(A)$ is very large, the problem is poorly conditioned and small relative residuals do not mean small relative errors.

If $\text{cond}(A)$ is not too large, the problem is well conditioned and a small relative residual is a reliable indicator of small relative error.

Note: Conditioning is a continuous spectrum. How large is "very large" depends on context.

Going back to example 9, $A = \begin{bmatrix} 0.780 & 0.563 \\ 0.913 & 0.656 \end{bmatrix}$.

Hence, $A^{-1} = \dfrac{1}{\det(A)} \begin{bmatrix} 0.656 & -0.563 \\ -0.913 & 0.780 \end{bmatrix}$

$$= 10^6 \begin{bmatrix} 0.656 & -0.563 \\ -0.913 & 0.780 \end{bmatrix}$$

Now, let's find $\text{Cond}(A)$.

$$\left.\begin{array}{l} \|A\|_\infty = 1.572 \\ \|A^{-1}\|_\infty = 1.693 \cdot 10^6 \end{array}\right\} \rightarrow \begin{array}{l} \text{cond}_\infty(A) = \|A\|_\infty \, \|A^{-1}\|_\infty \\ \qquad = 2.66 \cdot 10^6 \end{array}$$

$\dfrac{\|\bar{x} - \bar{\hat{x}}\|}{\|\bar{x}\|} \leq 2.66 \times 10^6 \dfrac{\|\bar{r}\|}{\|\bar{b}\|}$, meaning that the relative error in $\bar{x}$ could be as big as $2.66 \cdot 10^6$ times the relative residual.

This means that A is a poorly conditioned matrix and relative residual is not a reliable indicator of relative error.

7. Iterative Refinement:

- One way to improve $\bar{\hat{x}}$ is to improve the mantissa length.
- Suppose that you've already solved $(A+E)\bar{\hat{x}} = \bar{b}$ and you want to solve $A\bar{x} = \bar{b}$.

$$(A+E)\bar{\hat{x}} = \bar{b} \iff A\bar{\hat{x}} + \bar{r} = \bar{b}$$
$$\iff A\bar{\hat{x}} = \bar{b} - \bar{r}$$

Now we have

$A\bar{x} = \bar{b}$ (1)

$A\bar{\hat{x}} = \bar{b} - \bar{r}$ (2)

If we do (1) - (2), we get $A(\bar{x} - \bar{\hat{x}}) = \bar{r}$.

Let $\bar{z} = \bar{x} - \bar{\hat{x}}$

Now, I'll solve $A\bar{z} = \bar{r}$.

Furthermore, $\bar{x} = \bar{\hat{x}} + \bar{z}$. However, this is a fallacy because we can't get $\bar{z}$. Instead, we get $\bar{\hat{z}}$.

- Algorithim:

1. Compute $\hat{x}^{(0)}$ by solving $A\bar{x} = \bar{b}$ in a FPS.
2. For $i = 0, 1, 2, \dots$ until the soln is good enough:
3. Compute $\bar{r}^{(i)} = b - A\bar{\hat{x}}^{(i)}$
4. Solve $A\bar{z}^{(i)} = \bar{r}^{(i)}$ for some $\bar{z}^{(i)}$
5. Update $\bar{\hat{x}}^{(i+1)} = \bar{\hat{x}}^{(i)} + \bar{\hat{z}}^{(i)}$

8. Vector Norms:

- Let $\bar{x} \in R^n$. Then:

a. $$\|\bar{x}\|_1 = \sum_{i=1}^{n} |x_i|$$

b) $$\|\bar{x}\|_2 = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}$$

c) $$\|\bar{x}\|_\infty = \max_{1 \le i \le n} \left(|x_i|\right)$$

In general, for $p>0$, $$\|\bar{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$$

- Properties:

a) $\|\bar{x}\| > 0$, if $\bar{x} \neq \bar{0}$
b) $\|a\bar{x}\| = |a| \, \|\bar{x}\|$ for any scalar a.
c) $\|\bar{x}+\bar{y}\| \le \|\bar{x}\| + \|\bar{y}\|$ Triangle Inequality
d) $\|\bar{x}\|_1 \ge \|\bar{x}\|_2 \ge \|\bar{x}\|_\infty$
e) $\|\bar{x}\|_1 \le \sqrt{n} \, \|\bar{x}\|_2$
f) $\|\bar{x}\|_2 \le \sqrt{n} \, \|\bar{x}\|_\infty$
g) $\|\bar{x}\|_1 \le n \, \|\bar{x}\|_\infty$

E.g. 10 Let $\bar{x} = [3, 5, -7, 8]$

$$\|\bar{x}\|_1 = \sum_{i=1}^{n} |x_i|$$
$$= 3+5+7+8$$
$$= 23$$

$$\|\bar{x}\|_2 = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}$$
$$= (3^2+5^2+7^2+8^2)^{1/2}$$
$$= \sqrt{147}$$

$$\|\bar{x}\|_\infty = 8$$

9. Matrix Norms:

- Let $A \in \mathbb{R}^{n \times m}$ (I.e. A is a $n \times m$ matrix).

- $$\|A\|_1 = \max_{1 \le j \le m} \left(\sum_{i=1}^{n} |a_{ij}| \right)$$
$$= \text{Max absolute col sum}$$

- $$\|A\|_\infty = \max_{1 \le i \le n} \left(\sum_{j=1}^{m} |a_{ij}| \right)$$
$$= \text{Max absolute row sum}$$

E.g. 11 Let $A = \begin{bmatrix} 1 & -7 \\ -2 & -3 \end{bmatrix}$. Find $\|A\|_1$ and $\|A\|_\infty$

Soln:

$$\|A\|_1 = \max(1+|-2|,\ |-7|+|-3|)$$
$$= \max(3, 10)$$
$$= 10$$

$$\|A\|_\infty = \max(1+|-7|,\ |-2|+|-3|)$$
$$= \max(8, 5)$$
$$= 8$$

– Properties:

a) $\|A\| > 0$, if $A \neq 0$

b) $\|\alpha A\| = |\alpha| \, \|A\|$ for any scalar α

c) $\|A+B\| \le \|A\| + \|B\|$

d) $\|AB\| \le \|A\| \cdot \|B\|$

e) $\|A\bar{x}\| \le \|A\| \cdot \|\bar{x}\|$ for any vector $\bar{x}$.

f) In general, $\|A\| = \max\limits_{\bar{x} \neq \bar{0}} \dfrac{\|A\bar{x}\|}{\|\bar{x}\|}$

g) $\|A\|_2 = \sigma_{max}(A)$, where $\sigma_{max}(A)$ is the largest singular value of matrix A.

h) $\|A\|_2 \le \|A\|_F$, where $\|A\|_F = \left(\sum\limits_{i=1}^{m} \sum\limits_{j=1}^{n} |a_{ij}|^2 \right)^{1/2}$

and is called the Frobenius norm.

10. Tutorial Examples:

E.g. 12 Given $A = \begin{bmatrix} 2 & 6 & 6 \\ 3 & 5 & 12 \\ 6 & 6 & 12 \end{bmatrix}$ and $\bar{b} = \begin{bmatrix} 20 \\ 25 \\ 30 \end{bmatrix}$

use LU factorization to solve $A\bar{x} = \bar{b}$.

Soln:

We want $A = LU$.

Then, $A\bar{x} = \bar{b} \iff (LU)\bar{x} = \bar{b}$

$\iff L\bar{d} = b$, where $\bar{d} = U\bar{x}$

$L_1 = \begin{bmatrix} 1 & 0 & 0 \\ -3/2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$

$$L_1 A = \begin{bmatrix} 1 & 0 & 0 \\ -3/2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 6 & 6 \\ 3 & 5 & 12 \\ 6 & 6 & 12 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 6 & 6 \\ 0 & -4 & 3 \\ 0 & -12 & -6 \end{bmatrix}$$

$$L_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$$

$$L_2(L_1 A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 6 & 6 \\ 0 & -4 & 3 \\ 0 & -12 & -6 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 6 & 6 \\ 0 & -4 & 3 \\ 0 & 0 & -15 \end{bmatrix} \leftarrow u$$

$$u = \begin{bmatrix} 2 & 6 & 6 \\ 0 & -4 & 3 \\ 0 & 0 & -15 \end{bmatrix}$$

$$L = L_1^{-1} \cdot L_2^{-1}$$
$$= L_1^{-1} + L_2^{-1} - I$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ 3/2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 3/2 & 1 & 0 \\ 3 & 3 & 1 \end{bmatrix}$$

We now have L and U.
I will define $\bar{d} = U\bar{x}$.

$L\bar{d} = \bar{b}$, solve for $\bar{d}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 3/2 & 1 & 0 \\ 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} 20 \\ 25 \\ 30 \end{bmatrix}$$

$d_1 = 20$

$\frac{3}{2} d_1 + d_2 = 25 \Rightarrow 30 + d_2 = 25 \Rightarrow d_2 = -5$

$3d_1 + 3d_2 + d_3 = 30 \Rightarrow 60 + (-15) + d_3 = 30 \Rightarrow d_3 = -15$

$$\bar{d} = \begin{bmatrix} 20 \\ -5 \\ -15 \end{bmatrix}$$

Now, solve for $\bar{x}$ in $U\bar{x} = \bar{d}$

$$\begin{bmatrix} 2 & 6 & 6 \\ 0 & -4 & 3 \\ 0 & 0 & -15 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 20 \\ -5 \\ -15 \end{bmatrix}$$

$x_3 = 1$

$-4x_2 + 3x_3 = -5 \Rightarrow -4x_2 + 3 = -5 \Rightarrow -4x_2 = -8 \Rightarrow x_2 = 2$

$2x_1 + 6x_2 + 6x_3 = 20 \Rightarrow 2x_1 + 12 + 6 = 20 \Rightarrow 2x_1 = 2 \Rightarrow x_1 = 1$

$$\bar{x} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Note: I previously did this example on page 17 as example 7. You can see that we got the same soln.

E.g. 13 Do the same example as example 12, but this time, also do pivoting.

Soln:
Recall that we want the elements along the main diagonal, the pivot, to be the largest value in that column where the entries are chosen from and below the pivot.

I.e. The position of a pivot is a_{kk}, $1 \le k \le n$. We want a_{kk} to be the largest value in col k starting from row k and going down.

Looking at the first col of A, we see that the largest value of col 1 starting from row 1 is the 6 on row 3. Hence, we swap rows 1 and 3.

$$P_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$P_1 A = \begin{bmatrix} 6 & 6 & 12 \\ 3 & 5 & 12 \\ 2 & 6 & 6 \end{bmatrix}$$

$$L_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ -1/3 & 0 & 1 \end{bmatrix}$$

$$L_1(P_1 A) = \begin{bmatrix} 6 & 6 & 12 \\ 0 & 2 & 6 \\ 0 & 4 & 2 \end{bmatrix}$$

Now, looking at col 2 of $L_1(P_1 A)$, we see that the highest value of col 2 starting from row 2 is 4. So, we swap rows 2 and 3.

$$P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$P_2(L_1 P_1 A) = \begin{bmatrix} 6 & 6 & 12 \\ 0 & 4 & 2 \\ 0 & 2 & 6 \end{bmatrix}$$

$$L_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{2} & 1 \end{bmatrix}$$

$$L_2(P_2 L_1 P_1 A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 6 & 6 & 12 \\ 0 & 4 & 2 \\ 0 & 2 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 6 & 12 \\ 0 & 4 & 2 \\ 0 & 0 & 5 \end{bmatrix} \leftarrow U$$

$L_2 P_2 L_1 P_1 A \Longleftrightarrow L_2 \underbrace{P_2 L_1 P_2}_{\tilde{L}_1} P_2 P_1 A$, because $P_i \cdot P_i = I$

$$\tilde{L}_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1/3 & 1 & 0 \\ -1/2 & 0 & 1 \end{bmatrix}$$

Recall: If you do $P_2 M$, you switch the 2^{nd} and 3^{rd} rows of M. If you do $M P_2$, you switch the 2^{nd} and 3^{rd} cols of M.

$$L_2 \tilde{L}_1 P_2 P_1 A = U$$

$$\Longleftrightarrow \underbrace{P_2 P_1}_{P} A = \underbrace{\tilde{L}_1^{-1} L_2^{-1}}_{L} U$$

$$PA = LU$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ 1/2 & 1/2 & 1 \end{bmatrix} \begin{bmatrix} 6 & 6 & 12 \\ 0 & 4 & 2 \\ 0 & 0 & 5 \end{bmatrix}$$

Recall that we started off with $A\bar{x} = \bar{b}$.
Now, we have $PA\bar{x} = P\bar{b}$, where $P = P_2 P_1$

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$P\bar{b} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 20 \\ 25 \\ 30 \end{bmatrix}$$

$$= \begin{bmatrix} 30 \\ 20 \\ 25 \end{bmatrix}$$

$LU\bar{x} = P\bar{b}$
Let $\bar{d} = U\bar{x}$
I will solve $L\bar{d} = P\bar{b}$ for $\bar{d}$.
Then, I will solve $U\bar{x} = \bar{d}$ for $\bar{x}$.

$L\bar{d} = P\bar{b}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ 1/2 & 1/2 & 1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} 30 \\ 20 \\ 25 \end{bmatrix}$$

$d_1 = 30$

$\frac{d_1}{3} + d_2 = 20 \rightarrow 10 + d_2 = 20 \rightarrow d_2 = 10$

$\frac{d_1}{2} + \frac{d_2}{2} + d_3 = 25 \rightarrow 15 + 5 + d_3 = 25 \rightarrow d_3 = 5$

$$\bar{d} = \begin{bmatrix} \; \\ 10 \\ 5 \end{bmatrix} 30$$

$U\bar{x} = \bar{d}$

$$\begin{bmatrix} 6 & 6 & 12 \\ 0 & 4 & 2 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \; \\ 10 \\ 5 \end{bmatrix} 30$$

$x_3 = 1$

$4x_2 + 2x_3 = 10 \rightarrow 4x_2 + 2 = 10 \rightarrow 4x_2 = 8 \rightarrow x_2 = 2$

$6x_1 + 6x_2 + 12x_3 = \underset{30}{} \rightarrow 6x_1 + 12 + 12 = 30 \rightarrow 6x_1 = 6 \rightarrow x_1 = 1$

$$\bar{x} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

which is what we got in the previous 2 tries.